ON SOME NEW IDENTITIES FOR THE FIBONOMIAL COEFFICIENTS

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Abstract. Let F_n be the n-th Fibonacci number. The Fibonomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_F$ are defined for $n \geq k > 0$ as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_F = 0$ for $n < k$. In this paper, we shall provide several identities among Fibonomial coefficients. In particular, we prove that

$$\sum_{j=0}^{4l+1} \operatorname{sgn}(2l-j) \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n-j} = -\frac{F_{2l-1}}{F_{4l+1}} \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F F_{n-4l-1},$$

holds for all non-negative integers n and l.

1. Introduction

In 1915, Fontené published a one-page note [1] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence $\{a_n\}$ of real or complex numbers. Thus the generalized binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \frac{a_n a_{n-1} \cdots a_{n-k+1}}{a_1 a_2 \cdots a_k}.$$

Setting $a_n = n$ we recover the ordinary binomial coefficients, while $a_n = q^n - 1$ we obtain the q-binomial coefficients studied by Gauss, Euler, Cauchy and which were shortly called q-Gaussian coefficients (Gauss q-binomial coefficients). The sequence $\{a_n\}$ is essentially arbitrary but we do require that $a_n \neq 0$ for $n \geq 1$.

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The generalized binomial coefficients have many interesting properties. Obviously, for an integer $n \ge 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \begin{bmatrix} n \\ n-k \end{bmatrix}_a , \quad \begin{bmatrix} n \\ 1 \end{bmatrix}_a = a_n \text{ and } \begin{bmatrix} n \\ n \end{bmatrix}_a = 1 .$$

Specially, take a sequence $\{a_n\}$ of positive integers generated by

$$a_{n+2} = g\, a_{n+1} - h\, a_n \tag{1}$$

for integers $n \ge 0$, where g, $h \ne 0$ are real numbers. Let α and β, $\alpha \ne \beta$, be the roots of the characteristic equation $x^2 - gx + h = 0$ and let u_n, v_n be the solutions of (1) defined by $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $v_n = \alpha^n + \beta^n$. Jarden [3] found the following formula

$$\sum_{j=0}^{k} (-1)^j \begin{bmatrix} n \\ k \end{bmatrix}_a h^{\frac{j-1}{2}} z_{n+k-j} = 0 ,$$

where z_n is the product of the n-th terms of $k-1$ sequences satisfying (1).

Since 1964, there has been an accelerated interest in *Fibonomial coefficients*, which correspond to the choice $a_n = F_n$, where F_n are the Fibonacci numbers defined by (1) for $g = 1$, $h = -1$.

It is easy to write the key recurrence formula in the form

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = F_{k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + F_{n-k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F .$$

We refer the reader to recent papers [4] and [5] for results on the spacing and perfect powers among Fibonomial coefficients, respectively.

For the signed Fibonomial coefficients $(-1)^{\frac{j}{2}(j+1)} \left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right]_F$ (named A055870 in Sloane's encyclopedia of sequences), we can found the following identity, see [14], (a little rewritten)

$$\sum_{j=0}^{k} (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} k \\ j \end{bmatrix}_F F_{n-j}^{k-1} = 0 ,$$

where n, k are any positive integers with $n \ge k+1$.

In [6], we obtain for any non-negative integer $l \equiv 0, 2 \pmod 4$ the following identity

$$\sum_{j=0}^{l} (-1)^{\frac{j}{2}(j+(-1)^{l/2})} \begin{bmatrix} l \\ j \end{bmatrix}_F F_{n+l-j} = 0 .$$

In this paper, we shall study the similar sums of the Fibonomial coefficients, but for $l \equiv 1, 3 \pmod 4$. The following theorem is our main result.

Theorem 1. *Let l, n be any non-negative integers. Then*

$$\sum_{j=0}^{4l+3} sgn(2l+1-j) \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_{n-j} = \frac{F_{2l}}{F_{4l+3}} \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{n-4l-3} \tag{2}$$

and

$$\sum_{j=0}^{4l+1} sgn(2l-j) \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n-j} = -\frac{F_{2l-1}}{F_{4l+1}} \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F F_{n-4l-1} \ . \tag{3}$$

As usual, in the above statement, $sgn(x)$ denotes the *sign function* of x, defined by $sgn(0) = 0$ and $sgn(x) = x/|x|$, for $x \neq 0$.

2. The preliminary results

First, we extend the definition of the Fibonomial coefficients, for any integers n, k, by the following way

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{cases} 0, & \text{if } k < 0; \\ 1, & \text{if } k = 0 \ ; \\ \prod_{i=0}^{k-1} \frac{F_{n-i}}{F_{k-i}}, & \text{in other cases.} \end{cases}$$

Throughout what follows, $\{L_n\}$ denotes the sequence of *Lucas numbers* which follows the same recursive pattern as Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

Lemma 2. *Let k, l, m be any integers. Then*

$$F_{2l} - (-1)^l = L_{l-1}F_{l+1} \ , \tag{4}$$

$$F_{2l+1} - (-1)^l = L_{l+1}F_l \ , \tag{5}$$

$$F_{l+2}F_{2l} + (-1)^l F_{l-1} = F_{l+1}F_{2l+1}, \tag{6}$$

$$F_{k+1}F_{l-k} - F_kF_{l-k+1} = (-1)^k F_{l-2k}, \tag{7}$$

$$F_{l-k}F_{l-k+1} - F_{l+1}F_{l-2k} = (-1)^l F_kF_{k+1}. \tag{8}$$

$$F_{l+1}F_{2l+1} + (-1)^{l+1} F_{l-1} = F_{l+2}F_{2l}, \tag{9}$$

$$F_{2l+1}L_{l+1} + (-1)^l L_{l+2} = F_{l+2}L_l^2, \tag{10}$$

$$F_{k+1}F_{l-k+2} - F_{k+2}F_{l-k+1} = (-1)^k F_{l-2k}, \tag{11}$$

$$F^2_{m+k+l} + (-1)^{l+1} F^2_{m-k} = F_{2m+l}F_{2k+l}. \tag{12}$$

Proof. Identities (4) and (5) follows from identity

$$F_{n+m} - (-1)^m F_{n-m} = F_mL_n \ ,$$

see (15b) in [11], setting $m = l+1$, $n = l-1$ and $m = l$, $n = l+1$ respectively.

Identity (6) follows from identity

$$F_{a+b}F_{a+c} - F_aF_{a+b+c} = (-1)^a F_bF_c \ , \tag{13}$$

see (20a) in [11], setting $c = 1$, $a = l$, $b = l - 1$ and using basic recurrence.

Identity (7) follows from identity (13) setting $c = 1$, $a = k$, $b = l - 2k$.

Identity (8) follows from identity (13) setting $a = l + 1$, $b = k + 1$, $c = -k - 1$ and using identity $F_{-a} = (-1)^{a+1}F_a$ (see [11], identity (2)).

Identity (9) follows from identity (13) setting $a = l + 3$, $b = l - 1$, $c = 1$ and using Fibonacci recurrence relation.

Identity (10) can be rewritten by identities $L_{2n}+2\cdot(-1)^n = L_n^2$ and $F_n+L_n = 2F_{n+1}$ (see identities (17c) and (7b) in [11]), to the form

$$F_{l+2}L_{2l} - F_{2l+1}L_{l+1} = (-1)^l F_{l-1} \ ,$$

which follows from identity (13) setting $a = l + 1$, $b = l - 1$, $c = 1$.

Identity (11) follows from (7) using the basic recurrence.

Identity (12) follows identity (13) setting $c = m-p$, $b = m-p$ and $a = n+2p$.
■

Lemma 3. *Let l, $k \neq -1$ be any integers. Then the following holds*

$$(-1)^l \begin{bmatrix} l \\ k+1 \end{bmatrix}_F + \begin{bmatrix} l+1 \\ k \end{bmatrix}_F \frac{F_{2k-l}}{F_{k+1}} = \begin{bmatrix} l \\ k-1 \end{bmatrix}_F \ .$$

Proof. After overwriting the Fibonomial coefficients using their definition, we obtain the identity

$$(-1)^l F_{l-k+1}F_{l-k} + F_{2k-l}F_{l+1} = F_kF_{k+1} \ ,$$

which follows from identity (8) using again identity $F_{-a} = (-1)^{a+1}F_a$. ■

Lemma 4. *Let l, k be any integers. Then the following holds*

$$\begin{bmatrix} 4l+2 \\ 2k \end{bmatrix}_F \frac{F_{4l+2}F_{2l+3}}{F_{2l+1}} + \begin{bmatrix} 4l+3 \\ 2k+1 \end{bmatrix}_F (F_{4l-2k+3} - F_{2k+2})$$
$$+ \begin{bmatrix} 4l+3 \\ 2k+2 \end{bmatrix}_F (F_{4l-2k+2} - F_{2k+3}) = \begin{bmatrix} 4l+2 \\ 2k+2 \end{bmatrix}_F \frac{F_{4l+2}F_{2l+3}}{F_{2l+1}} \ .$$

Proof. After overwriting the Fibonomial coefficients using their definition we get

$$\begin{aligned} & F_{4l+2}F_{2l+3}F_{2k+1}F_{2k+2} + F_{4l+3}F_{2l+1}F_{2k+2}(F_{4l-2k+3} - F_{2k+2}) \\ & +F_{4l+3}F_{2l+1}F_{4l-2k+2}(F_{4l-2k+2} - F_{2k+3}) \\ = \ & F_{4l-2k+2}F_{4l-2k+1}F_{4l+2}F_{2l+3} \end{aligned}$$

which can be simplified by basic recurrence for the Fibonacci numbers and using identities (4), (7), where we replace k by $2k$ and l by $4l$, and (8), where we replace k by $2k + 1$ and l by $4l + 2$, to the form

$$F^2_{4l-2k+2} - F^2_{2k+2} = F_{4(l+1)}F_{4(l-k)}$$

which is the special form of identity (12) for $m = 0$, $n = 4(l - k)$ and $p = 2k + 2$.
■

Lemma 5. *Let $l \neq 0$, k be any integers. Then the following holds*

$$\begin{aligned} & \begin{bmatrix} 4l \\ 2k-1 \end{bmatrix}_F \frac{F_{4l}F_{2l+2}}{F_{2l}} + \begin{bmatrix} 4l+1 \\ 2k \end{bmatrix}_F (F_{4l-2k+2} - F_{2k+1}) \\ & + \begin{bmatrix} 4l+1 \\ 2k+1 \end{bmatrix}_F (F_{4l-2k+1} - F_{2k+2}) \\ = & \begin{bmatrix} 4l \\ 2k+1 \end{bmatrix}_F \frac{F_{4l}F_{2l+2}}{F_{2l}} \; . \end{aligned}$$

Proof. The proof is very similar as in Lemma 4. After overwriting the Fibonomial coefficients using their definition we get

$$\begin{aligned} & F_{4l}F_{2l+2}F_{2k}F_{2k+1} + F_{4l+1}F_{2l}F_{2k+1}(F_{4l-2k+2} - F_{2k+1}) \\ & + F_{4l+1}F_{2l}F_{4l-2k+1}(F_{4l-2k+1} - F_{2k+2}) \\ = & F_{4l-2k+1}F_{4l-2k}F_{4l}F_{2l+2} \; , \end{aligned}$$

which can be simplified by basic recurrence for the Fibonacci numbers and using identities (4), (11), where we replace k by $2k$ and l by $4l$, and (8), where we replace k by $2k$ and l by $4l$, to the form

$$F^2_{4l-2k+1} - F^2_{2k+1} = F_{2(2l+1)}F_{4(l-k)} \; ,$$

which is the special form of identity (12) for $m = 0$, $n = 4(l-k)$ and $p = 2k+1$. ■

Lemma 6. *Let l, k be any integers. Then*

$$(-1)^l \begin{bmatrix} l \\ k-1 \end{bmatrix}_F + \begin{bmatrix} l+1 \\ k+1 \end{bmatrix}_F \frac{F_{l-2k}}{F_{l-k+1}} = \begin{bmatrix} l \\ k+1 \end{bmatrix}_F$$

for $l \neq k-1$.

Proof. After overwriting the Fibonomial coefficients using their definition we obtain identity (8). ■

Theorem 7. *Let l be any integer and n any non-negative integer. Then*

$$\sum_{j=0}^{2n} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+2 \\ 2n \end{bmatrix}_F \frac{F_{4l+2}F_{2l+3}}{F_{2l+1}} \; . \tag{14}$$

Proof. We use the induction with respect to n. For $n = 0$ the assertion follows from (4) replacing l by $2l+2$ and using well-known equality $F_lL_l = F_{2l}$. Let us suppose that the identity holds for $n = k$ and prove it for $n = k+1$. The left-hand side can be rewritten as

$$\begin{aligned}\sum_{j=0}^{2k+2}\begin{bmatrix}4l+3\\ j\end{bmatrix}_F(F_{4l+4-j}-F_{j+1}) &= \sum_{j=0}^{2k}\begin{bmatrix}4l+3\\ j\end{bmatrix}_F(F_{4l+4-j}-F_{j+1})\\ &+\begin{bmatrix}4l+3\\ 2k+1\end{bmatrix}_F(F_{4l-2k+3}-F_{2k+2})\\ &+\begin{bmatrix}4l+3\\ 2k+2\end{bmatrix}_F(F_{4l-2k+2}-F_{2k+3})\end{aligned}$$

and the proof follows from Lemma 4. ■

Theorem 8. *Let l be any integer and n any positive integer. Then*

$$\sum_{j=1}^{n}\begin{bmatrix}4l-1\\ 2j-1\end{bmatrix}_F\frac{F_{4(l-j)}}{F_{2j}}=\begin{bmatrix}4l-2\\ 2n\end{bmatrix}_F-1\ . \tag{15}$$

Proof. We use the induction with respect to n. For $n=1$ the assertion is implied by (8) putting $k=1$ and replacing l by $4l-2$. Let us suppose that proved identity holds for $n=k$ and prove it for $n=k+1$. The left side of (15) has the form

$$\sum_{j=1}^{k+1}\begin{bmatrix}4l-1\\ 2j-1\end{bmatrix}_F\frac{F_{4(l-j)}}{F_{2j}}=\sum_{j=1}^{k}\begin{bmatrix}4l-1\\ 2j-1\end{bmatrix}_F\frac{F_{4(l-j)}}{F_{2j}}+\begin{bmatrix}4l-1\\ 2k+1\end{bmatrix}_F\frac{F_{4(l-k-1)}}{F_{2(k+1)}}$$

and the proved identity follows from Lemma 3 replacing k by $2k+1$ and l by $4l-2$. ■

Corollary 9. *Let n be any positive integer. Then*

$$\begin{aligned}\sum_{j=1}^{n}(-1)^jL_{2j} &= (-1)^nF_{2n+1}-1\ ,\\ \sum_{j=1}^{n}(-1)^jF_{2j+1}F_{2j+2}F_{2j+3}F_{4(j+1)} &= \frac{(-1)^n}{5}F_{2n+1}F_{2n+2}F_{2n+3}F_{2n+4}F_{2n+5}-6\ .\end{aligned}$$

Proof. We get the first relation setting $l=0$ in (15) and using identities $\begin{bmatrix}-2\\ 2n\end{bmatrix}_F=(-1)^nF_{2n+1}$ and $\begin{bmatrix}-1\\ 2j-1\end{bmatrix}_F=(-1)^{j+1}$, which follow from our extension of definition of the Fibonomial coefficient. The second relation we obtain similarly setting $l=-1$ in (15). ■

Theorem 10. *Let l and n be any integers. Then*

$$\begin{aligned}\sum_{j=1}^{n}\sigma(j,l) &= \begin{bmatrix}4l-2\\2n\end{bmatrix}_F - 1\ , \ \textit{where}\\ \sigma(j,l) &= \begin{cases} -1, & j=2l;\\ \begin{bmatrix}4l-1\\2j\end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2(2l-j)}}, & j\neq 2l,\end{cases}\end{aligned}$$

and

$$\sum_{j=0}^{n}\begin{bmatrix}4l\\2j\end{bmatrix}_F F_{4(l-j)} = F_{4l}\begin{bmatrix}4l-2\\2n\end{bmatrix}_F\ .$$

Proof. By overwriting the Fibonomial coefficients we obtain

$$\begin{bmatrix}4l-1\\2j-1\end{bmatrix}_F\frac{F_{4(l-j)}}{F_{2j}} = \begin{cases} -1, & j=2l;\\ \begin{bmatrix}4l-1\\2j\end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2(2l-j)}}, & j\neq 2l\end{cases}$$

and the first identity follows from (15). The second identity clearly holds for $l=0$ and for $l\neq 0$ we use the relation

$$\begin{bmatrix}4l-1\\2j-1\end{bmatrix}_F\frac{F_{4(l-j)}}{F_{2j}} = \begin{bmatrix}4l\\2j\end{bmatrix}_F\frac{F_{4(l-j)}}{F_{4l}},$$

which can be derived by overwriting the Fibonomial coefficients again, and the assertion is implied by (15). ∎

Theorem 11. *Let $l\neq 0$ be any integer and n any positive integer. Then*

$$\sum_{j=0}^{2n-1}\begin{bmatrix}4l+1\\j\end{bmatrix}_F (F_{4l+2-j}-F_{j+1}) = \begin{bmatrix}4l\\2n-1\end{bmatrix}_F\frac{F_{2l+2}F_{4l}}{F_{2l}}\ . \tag{16}$$

Proof. We use the induction with respect to n. For $n=1$ we obtain after overwriting the Fibonomial coefficients the following identity

$$F_{4l+2}-1+F_{4l+1}^2-F_{4l+1} = \frac{F_{4l}^2F_{2l+2}}{F_{2l}}\ .$$

Using relations $F_{2l}=F_lL_l$ and (5) we can rewrite it to the identity $L_{2l+2}+F_{4l+1}L_{2l+1}=L_{2l}^2F_{2l+2}$, which follows from (10) replacing l by $2l$.

Let us suppose, by the induction hypothesis, that the proved identity holds for $n = k$ and prove it for $n = k + 1$. The left side we can write as

$$\begin{aligned}
&\sum_{j=0}^{2k+1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) \\
= &\sum_{j=0}^{2k-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) + \begin{bmatrix} 4l+1 \\ 2k \end{bmatrix}_F (F_{4l-2k+2} - F_{2k+1}) \\
&+ \begin{bmatrix} 4l+1 \\ 2k+1 \end{bmatrix}_F (F_{4l-2k+1} - F_{2k+2})
\end{aligned}$$

and the proved identity follows from Lemma 5. ■

3. The proof of Theorem 1

Firstly we prove identity (2). This identity can be rewritten as

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_{n+4l+3-j} - \sum_{j=2l+2}^{4l+3} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_{n+4l+3-j} = \frac{F_{2l}}{F_{4l+3}} \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_n$$

and replacing j by $(4l+3) - j$ in the second sum and simplifying to the form

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{n+4l+3-j} - F_{n+j}) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F \left(F_{n+2l+1} + \frac{F_n F_{2l}}{F_{4l+3}} \right) .$$

We prove it by induction with respect to n. Let $n = 0$, thus we have to show that

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+3-j} - F_j) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{2l+1} .$$

But this identity follows from the relation $\begin{bmatrix} m \\ j \end{bmatrix}_F F_{m-j} = \begin{bmatrix} m \\ j+1 \end{bmatrix}_F F_{j+1}$, where m, j are any non-negative integers, which can be easily obtained overwriting the Fibonomial coefficients, by the following way

$$\begin{aligned}
\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+3-j} - F_j) &= \sum_{j=0}^{2l} \left(\begin{bmatrix} 4l+3 \\ j+1 \end{bmatrix}_F F_{j+1} - \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_j \right) \\
&= \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{2l+1} - \begin{bmatrix} 4l+3 \\ 0 \end{bmatrix}_F F_0 \\
&= \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{2l+1} .
\end{aligned}$$

Let $n = 1$. Then we have to prove the identity

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F \left(F_{2l+2} + \frac{F_{2l}}{F_{4l+3}} \right) .$$

It can be simplified with respect to (6), where we replace l by $2l+1$, to

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F \frac{F_{2l+3}F_{4l+2}}{F_{4l+3}}$$

and

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+2 \\ 2l \end{bmatrix}_F \frac{F_{4l+2}F_{2l+3}}{F_{2l+1}}$$

but this is a special case of identity (14) for $n = l$ only.

If we consider that (2) holds for n and $n+1$, then adding of appurtant identities and using the basic recurrence $F_{n+2} = F_n + F_{n+1}$ the proof of (2) is over.

Now we prove identity (3). It obviously holds for $l = 0$ therefore we consider $l > 0$ in the next part of proof.

The identity can be rewritten as

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n+4l+1-j} - \sum_{j=2l+1}^{4l+1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n+4l+1-j} = -\frac{F_{2l-1}}{F_{4l+1}} \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F F_n .$$

and replacing j by $(4l+1) - j$ in the second sum and simplifying to the form

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{n+4l+1-j} - F_{n+j}) = \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F \left(F_{n+2l} - \frac{F_n F_{2l-1}}{F_{4l+1}} \right) .$$

We again prove it by induction with respect to n. For $n = 0$ the proof can be done analogously as proof of (2) for $n = 1$.

Let $n = 1$. Then we have to show that

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) = \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F \left(F_{2l+1} - \frac{F_{2l-1}}{F_{4l+1}} \right) .$$

This relation can be simplified using (9), where we replace l by $2l$, to

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) = \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F \frac{F_{2l+2}F_{4l}}{F_{4l+1}} .$$

and

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) = \begin{bmatrix} 4l \\ 2l-1 \end{bmatrix}_F \frac{F_{2l+2}F_{4l}}{F_{2l}} \ ,$$

which is a special case of identity (16). The induction leap is the same as in the proof of identity (2).

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